102. Let $f \in C^3(R^n, R)$ with $f(0) = f'(0) = 0$. Prove that there exist $h \in C^3(R^m, S_n(R))$, such that $f(x) = x^t h(x) x$, when $S_n(R)$, is the set of symmetric matrix, and x^t is the transpose of x.

(Jozsef Wildt IMC 2016)

Solution by Moubinool Omarjee, Paris, France. We have $f(x) = x^t h(x) x$, where $h(x) = \int_0^1 (1-u)H(ux)du$ with $H(v) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(v)\right)$ the Hessian of f, h is of class C^1 with theorems of derivative under integral sign.

103. Find the nature of the series $\sum\limits_{n\geq 1} \frac{e^{i\ln(p_n)}}{p_n}$ when $(p_n)_{n\geq 1}$ is the prime number increasing order, and i imaginary complex number.

(Jozsef Wildt IMC 2016)

Solution We didnt receive any solution. The solutions for this problem can also be sent during this issue.

104. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 + \left(\frac{(6n+1)b-c}{n(c+a)}\right)^2 + \left(\frac{(6n+1)c-a}{n(a+b)}\right)^2 \geq 27$$

for any positive integer $n \geq 1$.

(Jozsef Wildt IMC 2016)

Solution 1 by Arkady Alt, San Jose, California, USA.
Since $x^2 + y^2 + z^2 \geq \dfrac{(x+y+z)^2}{3}$ for any real x, y, z we obtain

$$\sum_{cyc}\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{(6n+1)a-b}{n(b+c)}\right)^2.$$

Since $\sum\limits_{cyc}\dfrac{(6n+1)a-b}{n(b+c)} = 6\sum\limits_{cyc}\dfrac{a}{b+c} + \dfrac{1}{n}\sum\limits_{cyc}\dfrac{a-b}{b+c}$ and by Cauchy Inequality

$$\sum_{cyc}\frac{a}{b+c} = (a+b+c)\sum_{cyc}\frac{1}{b+c} - 3 = \frac{1}{2}\left(\sum_{cyc}(b+c)\cdot\sum_{cyc}\frac{1}{b+c}\right) - 3 \geq \frac{9}{2} - 3 = \frac{3}{2}$$

then $\sum\limits_{cyc}\dfrac{(6n+1)a-b}{n(b+c)} \geq 6\cdot\dfrac{3}{2} + \dfrac{1}{n}\sum\limits_{cyc}\dfrac{a-b}{b+c} = 9 + \dfrac{1}{n}\sum\limits_{cyc}\dfrac{a-b}{b+c}$.

Noting that triples (a, b, c) and $\left(\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}\right)$ agreed in order $\left((a-b)\left(\dfrac{1}{b+c} - \dfrac{1}{c+a}\right) = \dfrac{(a-b)^2}{(b+c)(c+a)} \geq 0\right)$ by Rearrangement Inequality we have

$\sum\limits_{cyc}\dfrac{a}{b+c} \geq \sum\limits_{cyc}\dfrac{b}{b+c}$ and, therefore, $\sum\limits_{cyc}\dfrac{a-b}{b+c} \geq 0$. Hence $\sum\limits_{cyc}\dfrac{(6n+1)a-b}{n(b+c)} \geq 9$ and we finally obtain

$$\sum_{cyc}\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\cdot 81 = 27.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. From $a^2+b^2+c^2 \geq (a+b+c)^2/3$ which holds true regardless the sign of a, b, c, we get

$$\sum_{\text{cyc}} \left(\frac{(6n+1)a-b}{n(b+c)} \right)^2 \geq \frac{1}{3} \left(\sum_{\text{cyc}} \frac{(6n+1)a-b}{n(b+c)} \right)^2$$

so we come to prove

$$\left(6 \sum_{\text{cyc}} \frac{a}{b+c} + \frac{1}{n} \sum_{\text{cyc}} \frac{a-b}{b+c} \right)^2 \geq 81$$

Now $\sum\limits_{\text{cyc}} \dfrac{a}{b+c} \geq \dfrac{3}{2}$ is the famous Nesbitt's inequality so it suffices to show that

$$\sum_{\text{cyc}} \frac{a-b}{b+c} \geq 0 \iff \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \qquad (1)$$

Let's suppose $a \geq b \geq c$. It follows that

$$\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}$$

The Rearrangement–inequality yields (1) being (a, b, c) and $\left(\dfrac{1}{b+c}, \dfrac{1}{a+c}, \dfrac{1}{a+b} \right)$ equally sorted.

t Let $a \geq c \geq b$. It follows that

$$\frac{1}{b+c} \geq \frac{1}{a+b} \geq \frac{1}{a+c}$$

Again the Rearrangement–inequality yields (1) being (a, c, b) and $\left(\dfrac{1}{b+c}, \dfrac{1}{a+b}, \dfrac{1}{a+c} \right)$ equally sorted.

So we have got

$$\left(6 \sum_{\text{cyc}} \frac{a}{b+c} + \frac{1}{n} \sum_{\text{cyc}} \frac{a-b}{b+c} \right)^2 \geq \left(6 \sum_{\text{cyc}} \frac{a}{b+c} + 0 \right)^2 \geq 36 \frac{9}{4} = 81$$

and this completes the proof.

Also solved by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA and Michel Bataille, Rouen, France and Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.